

Mathematics of Music

The musical scale used in western music originated with the ancient Greeks. Originally there were seven primary notes to a scale. This is called a *diatonic* scale and even non-musicians are familiar with it as do-re-me-fa-so-la-ti-do. This scale can be played with the white keys on a piano keyboard, starting with *C*. As you go through a diatonic scale it is eight steps from *do* back to *do* again. For this reason, this range of notes is called an *octave*.

Over time, five more notes were added to the western musical scale. This 12-note scale is called a *chromatic* scale. For a scale starting with *C*, the five extra notes are played on a piano keyboard by pressing the black keys.

Musical scales are tied closely to mathematics. You will use a computer-interfaced Microphone to record the waveform of the sound that is produced. The computer will also perform a mathematical analysis of the waveform called an FFT to determine the fundamental frequency of the sound. Your challenge is to measure the frequencies of all the notes of a chromatic scale and then to determine a mathematical pattern.

The frequencies of two notes played at the same time that sound good to our ears usually have a special mathematical relationship. In this lab, you will investigate the mathematics behind intervals that are commonly used in music.

For this lab, an electronic keyboard is recommended, but you can use other instruments if they are available.

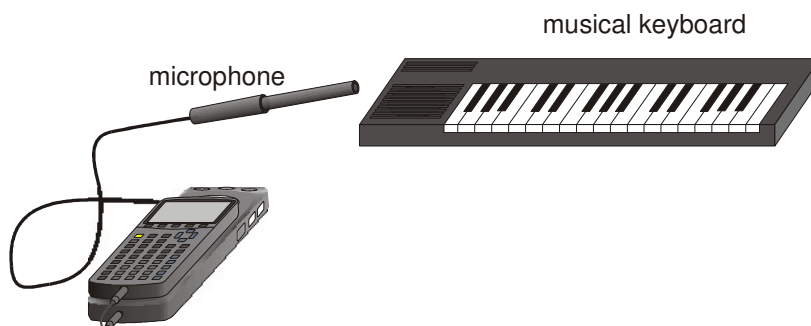


Figure 1

OBJECTIVES

- Determine the frequencies of the notes of a musical scale.
- Examine the differences and ratio between these notes.
- Determine the mathematical patterns used in musical scales.

MATERIALS


Power Macintosh or Windows PC
LabPro or Universal Lab Interface
Logger Pro

Vernier Microphone
electronic keyboard or other musical
instrument

PRELIMINARY QUESTIONS

1. What physics term is most closely related to what musicians refer to as *pitch*; that is, how high or low a musical note is?
2. What physics term is most closely related to the loudness of a musical note?
3. Make a hypothesis about how you think the frequencies of the notes of the diatonic (do-re-me...) scale are related. Do you think that each higher note is a certain number of Hz higher than the previous note? Would you guess that each succeeding note is higher by a constant ratio? What about the complete chromatic scale?

PROCEDURE

1. Connect the Vernier Microphone to Channel 1 of the LabPro or Universal Lab Interface.
2. Open the file in the Experiment 23 folder of *Physics with Computers*. The computer will display a graph window for displaying the waveform of the sound, a data table window, and an FFT window. An FFT is a Fast Fourier Transform, which is a mathematical analysis of a wave pattern broken into its component frequencies. We will use this window to determine the fundamental frequency of each tone we try.
3. Produce the first note to study, *C*. If you are using a keyboard, refer to Figure 2. This is middle *C* on a standard piano keyboard and the frequency should be approximately 262 Hz. Hold the Microphone close to the sound source and click  Collect. Hold the tone steady for a second or two (until the waveform appears on the screen). If you do not see a simple, well-defined wave pattern on the screen, try adjusting the Microphone position and try again.
4. Check the FFT window to see the frequency spectrum of the wave. This is a graph showing the relative amplitude of the frequencies present in the sound. Since we are using fairly simple sounds for this lab, you will probably see only one or two peaks on this graph. The lowest note present is called the *fundamental*. The frequency of the fundamental is displayed at the top left of the FFT window. Record this value in the data table.
5. Repeat Steps 3 and 4 for each of the notes listed in the Data Table. Notes 1 – 12 are the notes of one octave. Notes 13, 17, 20, and 25 are in higher octaves. Use the diagram as a reference point to locate the different notes on a keyboard. The musical notes are also listed in the table, with a subscript number indicating what octave they are in, such as C_4 .

DATA TABLE

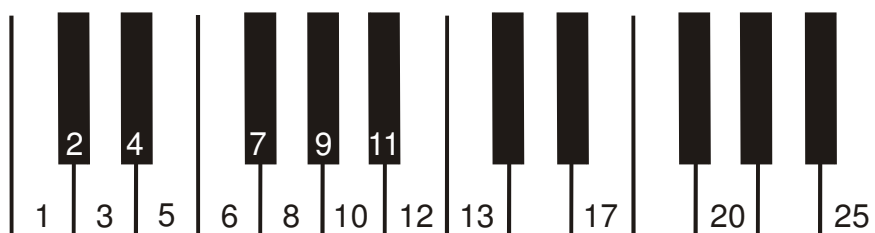


Figure 2

Key #	Note	Frequency (Hz)	Δf (Hz)	Ratio to previous note	Ratio to C_4
1	C₄				1.0
2	C ₄ [#]				
3	D₄				
4	E ₄ ^b				
5	E₄				
6	F₄				
7	F ₄ [#]				
8	G₄				
9	A ₄ ^b				
10	A₄				
11	B ₄ ^b				
12	B₄				
13	C₅				
17	E₅				
20	G₅				
25	C₆				

ANALYSIS

1. Calculate the difference in frequency between each note (except the first and the last three) to the previous one. Record your result in the column labeled Δf (Hz) in the data table. How consistent is this difference over the octave you studied?
2. Calculate the ratio of each frequency (except the first and the last three) to the previous one. Record your result in the data table. How consistent is this ratio over the octave you studied?
3. Fill in the unshaded cells of the last column by calculating the ratio of the frequency of the note to the frequency of the C_4 note.
4. Study your data on the white keys (the diatonic scale). These notes are listed in bold in the data table. Try to identify a pattern to these frequencies and ratios. Can the ratios in the last column be converted into ratios of small whole numbers? What would those whole-number ratios be? [Example: ratio = 1.33; small whole number ratio = 4:3]
5. Considering all the notes, what pattern exists between successive notes? Explain, citing specific evidence from your lab work.
6. Based on your data, predict the frequencies of the following notes: C_3 , C_7 , and D_5 .
7. The notes E_4 and C_4 represent a musical third and blend nicely when sounded together. The same is true for G_4 and C_4 , a musical fifth and C_5 and C_4 , a musical octave. What small, whole-number ratios characterize these musical intervals? Do you expect all thirds fifths and octaves to have the same ratios?

EXTENSIONS

1. Conduct the same measurements, using a different note as the starting point, rather than C₄. (Musicians would say this is “doing the lab in another key.”)
2. If you number the keys on a keyboard as in the diagram, you can construct a graph of frequency *vs.* key number. Do a curve fit on the data. What equation describes this graph? How do you interpret this equation?
3. Investigate the different types of tuning for musical instruments: just, equally tempered, Pythagorean, diatonic. Study the frequencies of a musical instrument that uses a completely different tuning system than the ones described here, such as a Southeast Asian instrument.